With regard to the two outstanding lines Keeler* has recorded that they are both stronger in α Orionis than in the solar spectrum.

The stellar behaviour of the iron flame lines of Group A is thus exactly in accord with their behaviour in the sunspot spectra as compared with the Fraunhöferic, and also just what would be expected from their laboratory behaviour.

The Theory of the Helmholtz Resonator.

By Lord Rayleigh, O.M., F.R.S.

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The ideal form of Helmholtz resonator is a cavernous space, almost enclosed by a thin, immovable wall, in which there is a small perforation establishing a communication between the interior and exterior gas. An approximate theory, based upon the supposition that the perforation is small, and consequently that the wave-length of the aërial vibration is great, is due to Helmholtz,† who arrived at definite results for perforations whose outline is circular or elliptic. A simplified, and in some respects generalised, treatment was given in my paper on "Resonance."‡ In the extreme case of a wave-length sufficiently great, the kinetic energy of the vibration is that of the gas near the mouth as it moves in and out, much as an incompressible fluid might do, and the potential energy is that of the almost uniform compressions and rarefactions of the gas in the interior. The latter is a question merely of the volume S of the cavity and of the quantity of gas which has passed, but the calculation of the kinetic energy presents difficulties which have been only partially overcome. In the case of simple apertures in the thin wall (regarded as plane), only circular and elliptic forms admit of complete treatment. The mathematical problem is the same as that of finding the electrostatic *capacity* of a thin conducting plate having the form of the aperture, and supposed to be situated in the open.

The project of a stricter treatment of the problem, in the case of a

* Quoted "Spectra of Stars of Secchi's Fourth Type," 'Pub. Yerkes Obs.,' vol. 2, p. 371 (1903).

† 'Crelle Jl. Math.,' vol. 57 (1860).

‡ 'Phil. Trans.,' vol. 161, p. 77 (1870); 'Scientific Papers,' vol. 1, p. 33. Also 'Theory of Sound,' ch. xvi.

spherical wall and an aperture of circular outline, has been in my mind more than 40 years, partly with the hope of reaching a closer approximation, and partly because some mathematicians have found the former method unsatisfactory, or, at any rate, difficult to follow. The present paper is on ordinary lines, using the appropriate spherical (Legendre's) functions, much as in a former one, "On the Acoustic Shadow of a Sphere."*

The first step is to find the velocity-potential (ψ) due to a normal motion at the surface of the sphere localised at a single point, the normal motion being zero at every other point. This problem must be solved both for the exterior and for the interior of the sphere, but in the end the potential is required only for points lying infinitely near the spherical surface. Then if we assume a normal motion given at every point on the aperture, that is on the portion of the spherical surface not occupied by the walls, we are in a position to calculate ψ upon the two sides of the aperture. If these values are equal at every point of the aperture, it will be a proof that the normal velocity has been rightly assumed, and a solution is arrived at. If the agreement is not sufficiently good—there is no question of more than an approximation—some other distribution of normal velocities must be tried. In what follows, the preliminary work is the same as in the paper last referred to, and the same notation is employed.

The general differential equation satisfied by ψ, and corresponding to a simple vibration, is

$$\frac{d^2\psi}{dx^2} + \frac{d^2\psi}{dy^2} + \frac{d^2\psi}{dz^2} + k^2\psi = 0, \tag{1}$$

where $k = 2\pi/\lambda$, and λ denotes the length of plane waves of the same pitch. For brevity we may omit k; it can always be restored on paying attention to "dimensions." The solution in polar co-ordinates applicable to a wave of the nth order in Laplace's series may be written (with omission of the time-factor)

$$\psi_n = S_n r^n \chi_n(r). \tag{2}$$

The differential equation satisfied by χ_n is

$$\frac{d^2\chi_n}{dr^2} + \frac{2n+2}{r}\frac{d\chi_n}{dr} + \chi_n = 0. \tag{3}$$

The solution of (3) applicable to a wave diverging outwards is

$$\chi_n(r) = \left(-\frac{d}{r\,dr}\right)^n \frac{e^{-ir}}{r}. \tag{4}$$

* 'Phil. Trans.,' A, vol. 203, p. 87 (1904); 'Scientific Papers,' vol. 5, p. 149.

Putting $n = 0$ and $n = 1$, we have

$$\chi_0(r) = \frac{e^{-ir}}{r}, \qquad \chi_1(r) = \frac{(1+ir)\,e^{-ir}}{r^3}. \tag{5}$$

It is easy to verify that (4) satisfies (3). For if χ_n satisfies (3), $r^{-1}\chi_n'$ satisfies the corresponding equation for χ_{n+1}. And $r^{-1}e^{-ir}$ satisfies (3) when $n = 0$.

From (3) and (4) the following sequence formulæ may be verified:

$$\chi_n'(r) = -r\chi_{n+1}(r), \tag{6}$$

$$r\chi_n'(r)+(2n+1)\,\chi_n(r) = \chi_{n-1}(r), \tag{7}$$

$$\chi_{n+1}(r) = \frac{(2n+1)\chi_n(r)-\chi_{n-1}(r)}{r^2}. \tag{8}$$

By means of the last, χ_2, χ_3, etc., may be built up in succession from χ_2 and χ_1.

From (2)

$$d\psi_n/dr = \mathrm{S}_n\,(nr^{n-1}\chi_n+r^2\chi_n'),$$

or with use of (7)

$$d\psi_n/dr = r^{n-1}\mathrm{S}_n\{\chi_{n-1}-(n+1)\chi_n\}. \tag{9}$$

Thus if U_n be the nth component of the normal velocity at the surface of the sphere ($r = c$)

$$\mathrm{U}_n = c^{n-1}\mathrm{S}_n\{\chi_{n-1}(c)-(n+1)\chi_n(c)\}. \tag{10}$$

When $n = 0$,

$$\mathrm{U}_0 = \mathrm{S}_0\chi_0'(c) = -\mathrm{S}_0 c\chi_1(c). \tag{11}$$

The introduction of S_n from (10), (11) into (2) gives ψ_n in terms of U_n supposed known.

When r is very great in comparison with the wave-length, we get from (4)

$$\chi_n(r) = \frac{i^n e^{-ir}}{r^{n+1}}, \tag{12}$$

so that

$$\psi_n = \mathrm{S}_n\,\frac{i^n e^{-ir}}{r}. \tag{13}$$

We have now to apply these formulæ to the particular case where U is sensible over an infinitesimal area $d\sigma$, but vanishes over the remainder of the surface of the sphere. If μ be the cosine of the angle (θ) between $d\sigma$ and the point at which U is expressed, $\mathrm{P}_n(\mu)$ Legendre's function, we have

$$\mathrm{U}_n = \frac{2n+1}{4\pi c^2}\,\mathrm{U}d\sigma\,.\,\mathrm{P}_n(\mu), \tag{14}$$

and accordingly for the velocity-potential at the *surface of the sphere*,

$$\psi = \frac{\mathrm{U}d\sigma}{4\pi c}\,\Sigma\,\frac{(2n+1)\chi_n(c)\,.\,\mathrm{P}_n(\mu)}{\chi_{n-1}(c)-(n+1)\chi_n(c)}. \tag{15}$$

When $n = 0$, $\chi_{n-1} - (n+1)\chi_n$ is to be replaced by $-c^2\chi_1$. Equation (15) gives the value of ψ at a point whose angular distance (θ) from $d\sigma$ is $\cos^{-1}\mu$. If χ_n has the form given by (4), the result applies to the *exterior* surface of the sphere.

We have also to consider the corresponding problem for the interior. The only change required is to replace χ_n as given in (4) by the form appropriate to the interior. For this purpose we might take simply the imaginary part of (4), but since a constant multiplier has no significance, it suffices to make

$$\chi_n(r) = \left(-\frac{d}{r\,dr}\right)\frac{\sin r}{r}. \tag{16}$$

With this alteration (15) holds good for the interior, U denoting the localised normal velocity at the surface, still measured outwards, since $U = d\psi/dr$.

We have now to introduce approximate values of $\chi_{n-1}(c) \div \chi_n(c)$ in (15), having regard to the assumed smallness of c, or rather kc. For this purpose we expand the sine and cosine of c:—

$$\frac{\cos c}{c} = \frac{1}{c} - \frac{c}{1.2} + \frac{c^3}{4!} - \frac{c^5}{6!} + \ldots,$$

$$-\frac{1}{c}\frac{d}{dc}\left(\frac{\cos c}{c}\right) = \frac{1}{c^3} - \frac{3c}{4!} + \frac{5c^3}{6!} - \frac{7c^5}{8!} + \ldots,$$

$$\left(-\frac{1}{c}\frac{d}{dc}\right)^2\frac{\cos c}{c} = \frac{3}{c^5} - \frac{5.3.c}{6!} + \frac{7.5.c^3}{8!} - \ldots,$$

and so on;

$$\frac{\sin c}{c} = 1 - \frac{c^2}{2.3} + \frac{c^4}{5!} - \frac{c^6}{7!} + \ldots,$$

$$-\frac{1}{c}\frac{d}{dc}\frac{\sin c}{c} = \frac{2}{2.3} - \frac{4c^2}{5!} + \frac{6c^4}{7!} - \ldots,$$

$$\left(-\frac{1}{c}\frac{d}{dc}\right)^2\frac{\sin c}{c} = \frac{4.2}{5!} - \frac{6.4.c^2}{7!} + \frac{8.6.c^4}{9!} - \ldots,$$

and so on. Thus for the outside

$$\chi_n = \frac{1.3.5\ldots(2n-1)}{c^{2n+1}} \times \left\{1 - \frac{c^{2n+2}}{2n!\,(2n+2)} - \frac{i\,c^{2n+1}}{1^2.3^2.5^2\ldots(2n-1)^2(2n+1)}\right\}. \tag{17}$$

For general values of n, we may take

$$\chi_{n-1} \div \chi_n = \frac{c^2}{2n-1}. \tag{18}$$

For $n = 1$

$$\frac{\chi_0}{\chi_1} = \frac{\frac{1}{c}-\frac{c}{2}-i}{\frac{1}{c^3}-\frac{c}{2.4}-\frac{i}{3}} = c^2\left(1-\frac{c^2}{2}-ic\right). \tag{19}$$

For $n = 2$

$$\frac{\chi_1}{\chi_2} = \frac{c^2}{3} + \text{terms in } c^5. \tag{20}$$

Thus in general by (18)

$$\frac{2n+1}{\chi_{n-1}/\chi_n-n-1} = -2+\frac{1}{n+1}-\frac{(2n+1)c^2}{(n+1)^2(2n-1)}; \tag{21}$$

while for $n = 1$

$$\frac{1}{\chi_0/\chi_1-2} = -2+\tfrac{1}{2}-\tfrac{3}{4}c^2+\text{terms in } c^3, \tag{22}$$

in accordance with (21). When $n = 0$

$$\frac{\chi_0}{-c^2\chi_1} = -1+\tfrac{1}{2}c^2+ic+\text{terms in } c^3. \tag{23}$$

Using these values in (15), we see that so far as c^2 inclusive

$$\begin{aligned}\Sigma\,(\text{outside}) &= (-1+\tfrac{1}{2}c^2+ic)\mathrm{P}_0 \\ &\quad+\left(-2+\frac{1}{2}-\frac{3c^2}{4}\right)\mathrm{P}_1+\left(-2+\frac{1}{3}-\frac{5c^2}{3^2.3}\right)\mathrm{P}_2+\ldots \\ &= -2\{\mathrm{P}_0(\mu)+\mathrm{P}_1(\mu)+\ldots+\mathrm{P}_n(\mu)+\ldots\} \\ &\quad+\left\{\mathrm{P}_0(\mu)+\frac{1}{2}\mathrm{P}_1(\mu)+\ldots+\frac{1}{n+1}\mathrm{P}_n(\mu)+\ldots\right\} \\ &\quad+ic+\tfrac{1}{2}c^2-\sum_1^\infty\frac{(2n+1)c^2}{(n+1)^2(2n-1)}\mathrm{P}_n(\mu).\end{aligned} \tag{24}$$

In like manner for the form of χ_n appropriate to the inside

$$\chi_n(c) = \frac{1}{1.3.5\ldots(2n+1)}\left\{1-\frac{c^2}{2(2n+3)}\right\}, \tag{25}$$

so that in general

$$\frac{\chi_{n-1}}{\chi_n} = 2n+1-\frac{c^2}{2n+3}, \tag{26}$$

and

$$\frac{2n+1}{\chi_{n-1}/\chi_n-n-1} = 2+\frac{1}{n}+\frac{(2n+1)c^2}{n^2(2n-3)}. \tag{27}$$

This suffices for $n = 1$ and onwards. When $n = 0$

$$\frac{\chi_0}{-c^2\chi_1} = -\frac{3}{c^2}\left\{1-\frac{c^2}{15}-\frac{c^4}{525}\right\}. \tag{28}$$

Accordingly, so far as c^2 inclusive,

$$\Sigma\,(\text{inside}) = 2\{P_0(\mu)+P_1(\mu)+\ldots+P_n(\mu)\}$$
$$+\left\{P_1(\mu)+\frac{1}{2}P_2(\mu)+\ldots+\frac{1}{n}P_n(\mu)\right\}$$
$$-\frac{3}{c^2}-1\frac{4}{5}+\frac{c^2}{175}+\sum_1^\infty\frac{(2n+1)c^2}{n^2(2n+3)}P_n(\mu). \tag{29}$$

The first two series of P's on the right of (24) and (29) become divergent when $\mu = 1$, or $\theta = 0$. To evaluate them we have

$$\frac{1}{\sqrt{\{1-2\alpha\cos\theta+\alpha^2\}}} = 1+\alpha P_1+\alpha^2 P_2+\ldots\,, \tag{30}$$

so that $$1+P_1+P_2+\ldots = \frac{1}{\sqrt{(2-2\cos\theta)}} = \frac{1}{2\sin\frac{1}{2}\theta}. \tag{31}$$

Again, by integration of (30),

$$\alpha+\tfrac{1}{2}\alpha^2P_1+\tfrac{1}{3}\alpha^3P_2+\ldots = \int_0^\alpha\frac{d\alpha}{\sqrt{\{1-2\alpha\cos\theta+\alpha^2\}}}$$
$$= \log\left[\alpha-\cos\theta+\sqrt{\{1-2\alpha\cos\theta+\alpha^2\}}\right]-\log[1-\cos\theta],\text{*}$$

so that $$1+\tfrac{1}{2}P_1+\tfrac{1}{3}P_2+\ldots = \log(1+\sin\tfrac{1}{2}\theta)-\log\sin\tfrac{1}{2}\theta. \tag{32}$$

In much the same way we may sum the third series $\Sigma n^{-1}P_n$. We have

$$P_1+\alpha P_2+\alpha^2P_3+\ldots = \frac{1}{\alpha\sqrt{\{1-2\alpha\mu+\alpha^2\}}}-\frac{1}{\alpha},$$

$$\alpha P_1+\tfrac{1}{2}\alpha^2P_2+\tfrac{1}{3}\alpha^3P_3+\ldots = \int_0^\alpha\frac{d\alpha}{\alpha\sqrt{\{1-2\alpha\mu+\alpha^2\}}}-\int_0^\alpha\frac{d\alpha}{\alpha}.$$

We denote the right-hand member of this equation by I and differentiate it with respect to μ.

Thus

$$\frac{dI}{d\mu} = \int_0^\alpha\frac{d\alpha}{\{(\alpha-\mu)^2+1-\mu^2\}^{3/2}} = \frac{\alpha-\mu}{(1-\mu^2)\sqrt{\{1-2\alpha\mu+\alpha^2\}}}+\frac{\mu}{1-\mu^2},$$

or when $\alpha = 1$

$$\frac{dI}{d\mu} = \frac{1}{4\sin\frac{1}{2}\theta\,.\,\cos^2\frac{1}{2}\theta}+\frac{\mu}{1-\mu^2} \tag{33}$$

On integration

$$I = \log\tan\tfrac{1}{4}(\pi-\theta)-\log\sin\theta+C. \tag{34}$$

* If we integrate this equation again with respect to a between the limits 0 and 1, we find

$$\frac{1}{1\,.\,2}+\frac{P_1}{2\,.\,3}+\ldots+\frac{P_n}{(n+1)(n+2)} = 1-2\sin\tfrac{1}{2}\theta+2\sin^2\tfrac{1}{2}\theta\left[\log(1+\sin\tfrac{1}{2}\theta)-\log\sin\tfrac{1}{2}\theta\right].$$

When θ is small, the more important part is

$$1-\theta-\tfrac{1}{2}\theta^2\log\theta.$$

The constant is to be found by putting $\mu = 0$, $\theta = \frac{1}{2}\pi$. In this case

$$\mathrm{I} = \int_0^\alpha \frac{d\alpha}{\alpha\sqrt{(1+\alpha^2)}} - \int_0^\alpha \frac{d\alpha}{\alpha} = \log 2 - \log\{1+\sqrt{(1+\alpha^2)}\}.$$

Thus
$$\mathrm{C} = \log\frac{2}{1+\sqrt{2}} - \log\tan\frac{\pi}{8} = \log 2,$$

and accordingly

$$\mathrm{P}_1 + \tfrac{1}{2}\mathrm{P}_2 + \tfrac{1}{3}\mathrm{P}_3 + \ldots = \log\tan\tfrac{1}{4}(\pi-\theta) - \log(\tfrac{1}{2}\sin\theta). \tag{35}$$

For the values of Σ in (15) we now have with restoration of k

$$\Sigma\,(\text{outside}) = -\frac{1}{\sin\frac{1}{2}\theta} - \log\sin\tfrac{1}{2}\theta + \log(1+\sin\tfrac{1}{2}\theta)$$
$$+ikc + \tfrac{1}{2}k^2c^2 - \sum_1^\infty \frac{(2n+1)\,k^2c^2}{(n+1)^2(2n-1)}\mathrm{P}_n(\mu), \tag{36}$$

$$\Sigma\,(\text{inside}) = \frac{1}{\sin\frac{1}{2}\theta} - \log(\tfrac{1}{2}\sin\theta) + \log\tan\tfrac{1}{4}(\pi-\theta)$$
$$-\frac{3}{k^2c^2} - \frac{9}{5} + \frac{k^2c^2}{175} + \sum_1^\infty \frac{(2n+1)\,k^2c^2}{n^2(2n+3)}\mathrm{P}_n(\mu). \tag{37}$$

These equations give the value of ψ at any point of the sphere, either inside or outside, due to a normal velocity at a single point, so far as k^2c^2 inclusive. The inside value is dominated by the term $-3/k^2c^2$, except when θ is small. As to the sums in k^2c^2 not evaluated, we may remark that they cannot exceed the values assumed when $\theta = 0$ and $\mathrm{P}_n(\mu) = 1$. Approximate calculation of the limiting values is easy. Thus

$$\sum_1^\infty \frac{2n+1}{(n+1)^2(2n-1)}$$
$$= \sum_1^5 \frac{2n+1}{(n+1)^2(2n-1)} - \sum_1^5\{n^{-2}-n^{-3}+\tfrac{3}{2}n^{-4}\} + \sum_1^\infty\{n^{-2}-n^{-3}+\tfrac{3}{2}n^{-4}\}$$
$$= -0{\cdot}79040 + 1{\cdot}64493 - 1{\cdot}20206 + 1{\cdot}62348 = 1{\cdot}2759.*$$

In like manner

$$\sum_1^\infty \frac{2n+1}{n^2(2n+3)} = -0{\cdot}9485 + \sum_1^\infty\{n^{-2}-n^{-3}+\tfrac{3}{2}n^{-4}\} = 1{\cdot}1178.$$

Our special purpose is concerned with the *difference* in the values of ψ on the two sides of the surface $r = c$, and thus only with the difference of Σ's. We have

$$\Sigma\,(\text{inside}) - \Sigma\,(\text{outside}) = \frac{2}{\sin\frac{1}{2}\theta} - \log\cos\frac{\theta}{2} + \log\frac{1-\tan\frac{1}{4}\theta}{1+\tan\frac{1}{4}\theta}$$
$$-\log\left(1+\sin\frac{\theta}{2}\right) - \frac{3}{k^2c^2} - \frac{9}{5} - ikc$$
$$+k^2c^2\left\{\frac{1}{175} - \frac{1}{2} + \sum_1^\infty \frac{(2n+1)\mathrm{P}_n(\mu)}{n^2(2n+3)} + \sum_1^\infty \frac{(2n+1)\mathrm{P}_n(\mu)}{(n+1)^2(2n+1)}\right\}. \tag{38}$$

* 'Chrystal's Algebra,' Part II, p. 343.

In the application we have to deal only with small values of θ and we shall omit k^2c^2, so that we take

$$\Sigma\,(\text{in}) - \Sigma\,(\text{out}) = \frac{2}{\sin\frac{1}{2}\theta} - \theta - \frac{3}{k^2c^2} - \frac{9}{5} - ikc\,; \tag{39}$$

it will indeed appear later that we do not need even the term in θ, since it is of the order k^2c^2.

In pursuance of our plan we have now to assume a form for U over the circular aperture and examine how far it leads to agreement in the values of ψ on the inside and on the outside. For this purpose we avail ourselves of information derived from the first approximation. If C, fig. 1, be the centre and CA the angular radius of the spherical segment constituting the aperture, P any other point on it, we assume that U at P is proportional to $\{\mathrm{CA}^2 - \mathrm{CP}^2\}^{-\frac{1}{2}}$, and we require to examine the consequences at another arbitrary point O.

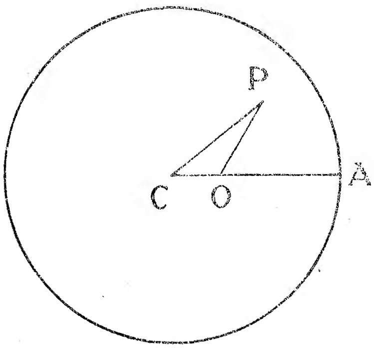

Fig. 1.

Writing CA $= a$, CO $= b$, PO $= \theta$, POA $= \phi$, we have from the spherical triangle

$$\cos \mathrm{CP} = \cos b \cos\theta + \sin b \sin\theta \cos\phi,$$

or when we neglect higher powers than the cube of the small angles,

$$\mathrm{CP}^2 = b^2 + \theta^2 + 2b\theta\cos\phi. \tag{40}$$

Thus

$$\mathrm{CA}^2 - \mathrm{CP}^2 = a^2 - b^2 - \theta^2 - 2b\theta\cos\phi = a^2 - b^2\sin^2\phi - (\theta + b\cos\phi)^2, \tag{41}$$

and we wish to make

$$\iint \frac{\sin\theta\, d\theta\, d\phi\,[\Sigma\,(\text{in}) - \Sigma\,(\text{out})]}{\sqrt{\{a^2 - b^2 - \theta^2 - 2b\theta\cos\phi\}}} = 0. \tag{42}$$

as far as possible for all values of b, the integration covering the whole area of aperture. We may write θ for $\sin\theta$, since we are content to neglect terms of order θ^2 in comparison with the principal term. Reference to (39) shows that as regards the numerator of the integrand we have to deal with terms in θ^0, θ^1, and θ^2.

For the principal term we have

$$4\iint \frac{d\theta\, d\phi}{\sqrt{\{a^2 - b^2 - \theta^2 - 2b\theta\cos\phi\}}}. \tag{43}$$

Now $$\int\frac{d\theta}{\sqrt{\{\quad\}}}=\int\frac{d(\theta+b\cos\phi)}{\sqrt{\{\quad\}}}=\sin^{-1}\frac{\theta+b\cos\phi}{\sqrt{\{a^2-b^2\sin^2\phi\}}}.$$

For a given ϕ the lower limit of θ is 0 and the upper limit θ_1 is such as to make $a^2=b^2+\theta_1{}^2+2b\theta_1\cos\phi$,

or $$\theta_1+b\cos\phi=\sqrt{(a^2-b^2\sin^2\phi)}. \tag{44}$$

Thus $$\int_0^{\theta_1}\frac{d\theta}{\sqrt{\{\quad\}}}=\frac{\pi}{2}-\sin^{-1}\frac{b\cos\phi}{\sqrt{(a^2-b^2\sin^2\phi)}}. \tag{45}$$

When this is integrated with respect to ϕ, the second part disappears, and we are left with π^2 simply, so that the principal term (43) is $4\pi^2$. That this should turn out independent of b, that is the same at all points of the aperture, is only what was to be expected from the known theory respecting the motion of an incompressible fluid.

The term in θ, corresponding to the constant part of $\Sigma\,(\text{in})-\Sigma\,(\text{out})$, is represented by

$$\iint\frac{\theta\,d\theta\,d\phi}{\sqrt{\{a^2-\mathrm{CP}^2\}}}. \tag{46}$$

Here $\theta\,d\theta\,d\phi$ is merely the polar element of area, and the integral is, of course, independent of b. To find its value we may take the centre C as the pole of θ. We get at once

$$2\pi\int_0^a\frac{\theta\,d\theta}{\sqrt{(a^2-\theta^2)}}=2\pi a\,; \tag{47}$$

so that this part of (42) is

$$-2\pi a\left(\frac{3}{k^2c^2}+\frac{9}{5}+ikc\right). \tag{48}$$

For the third part (in θ^2), we write

$$\theta^2=-(a^2-b^2-2b\theta\cos\phi-\theta^2)-2b\cos\phi\,(\theta+b\cos\phi)+a^2-b^2+2b^2\cos^2\phi,$$

giving rise to three integrals in θ, of which the first is

$$-\int d\theta\sqrt{\{a^2-b^2-2b\cos\phi-\theta^2\}}$$
$$=-\tfrac{1}{2}(\theta+b\cos\phi)\sqrt{\{a^2-b^2\sin^2\phi-(\theta+b\cos\phi)^2\}}$$
$$-\frac{a^2-b^2\sin^2\phi}{2}\sin^{-1}\frac{\theta+b\cos\phi}{\sqrt{(a^2-b^2\sin^2\phi)}}. \tag{49}$$

The second integral is

$$-2b\cos\phi\int\frac{(\theta+b\cos\phi)d\theta}{\sqrt{\{a^2-\mathrm{CP}^2\}}}=2b\cos\phi\sqrt{\{a^2-b^2-2b\theta\cos\phi-\theta^2\}}, \tag{50}$$

and the third is, as for the principal term,

$$(a^2-b^2+2b^2\cos^2\phi)\sin^{-1}\frac{\theta+b\cos\phi}{\sqrt{(a^2-b^2\sin^2\phi)}}. \tag{51}$$

Thus altogether when the three integrals are taken between the limits 0 and θ_1, we get

$$-\tfrac{3}{2}b\cos\phi\sqrt{(a^2-b^2)}+[\tfrac{1}{2}a^2+b^2(2\cos^2\phi+\tfrac{1}{2}\sin^2\phi-1)]$$
$$\times\left[\frac{\pi}{2}-\sin^{-1}\frac{b\cos\phi}{\sqrt{(a^2-b^2\sin^2\phi)}}\right],$$

and finally after integration with respect to ϕ

$$\tfrac{1}{2}\pi^2(a^2+\tfrac{1}{2}b^2). \tag{52}$$

Thus altogether the integral on the left of (42) becomes

$$4\pi^2-2\pi a\left(\frac{3}{k^2c^2}+\frac{9}{5}+ikc\right)-\frac{\pi^2}{2}\left(a^2+\frac{b^2}{2}\right). \tag{53}$$

In consequence of the occurrence of b^2, this expression cannot be made to vanish at all points of the aperture, a sign that the assumed form of U is imperfect. If, however, we neglect the last term, arising from $-\theta$ in $\Sigma\,(\text{in})-\Sigma\,(\text{out})$, our expression vanishes provided

$$\frac{3}{k^2c^2}+\frac{9}{5}+ikc=\frac{2\pi}{a}, \tag{54}$$

showing that a is of the order k^2c^2, so that this equation gives the relation between a and kc to a sufficient approximation. Helmholtz's solution corresponds to the neglect of the second and third terms on the left of (54), making

$$\frac{3}{k^2c^2}=\frac{2\pi}{a}=\frac{2\pi c}{\mathrm{R}}, \tag{55}$$

where R denotes the linear radius of the circular aperture. If we introduce $\lambda\,(=2\pi/k)$,

$$\lambda=\pi\sqrt{(2\mathrm{S}/\mathrm{R})}, \tag{56}$$

S denoting the capacity of the sphere, the known approximate value.

The third term on the left of (54) represents the decay of the vibration due to the propagation of energy away from the resonator. Omitting this for the moment, we have as the corrected value of λ,

$$\lambda=\pi\sqrt{(2\mathrm{S}/\mathrm{R})}\,.\left\{1-\frac{9}{10}\frac{\mathrm{R}}{2\pi c}\right\}. \tag{57}$$

Let us now consider the term representing decay of the vibrations. The time factor, hitherto omitted, is $e^{ik\mathrm{V}t}$, or if we take $k=k_1+ik_2$, $e^{-k_2\mathrm{V}t}\,e^{ik_1\mathrm{V}t}$. If $t=\tau$, the period, $k_1\mathrm{V}\tau=2\pi$, and $e^{-k_2\mathrm{V}\tau}=e^{-2\pi k_2/k_1}$. This is the factor by which the amplitude of vibration is reduced in one period. Now from (55)

$$kc=\sqrt{\left(\frac{3\mathrm{R}}{2\pi c}\right)},$$

so that (54) becomes

$$\frac{3}{k^2c^2} + i\sqrt{\left(\frac{3\mathrm{R}}{2\pi c}\right)} = \frac{2\pi c}{\mathrm{R}}, \tag{58}$$

whence

$$k_1 + ik_2 = \sqrt{\left(\frac{3\mathrm{R}}{2\pi c^3}\right)}\left\{1 + i\frac{\sqrt{3}}{2}\left(\frac{\mathrm{R}}{2\pi c}\right)^{3/2}\right\}, \tag{59}$$

and

$$\frac{2\pi k_2}{k_1} = \pi\sqrt{3}\left(\frac{\mathrm{R}}{2\pi c}\right)^{3/2}. \tag{60}$$

This gives the reduction of amplitude after one vibration. The decay is least when R is small relatively to c, although it is then estimated for a longer time.

The value found in (60) differs a little from that given in 'Theory of Sound,' § 311, where the aperture is supposed to be surrounded by an infinite flange, the effect of which is to favour the propagation of energy away from the resonator.

So far we have supposed the boundary of the aperture to be circular. A comparison with the corresponding process in 'Theory of Sound,' § 306 (after Helmholtz), shows that to the degree of approximation here attained the results may be extended to an elliptic aperture provided we replace R by

$$\frac{\pi \mathrm{R}_1}{2\mathrm{F}(e)}, \tag{61}$$

where R_1 denotes the semi-axis major of the ellipse, e the eccentricity, and F the symbol of the complete elliptic function of the first order. It is there further shown that for any form of aperture not too elongated, the truth is approximately represented if we take $\sqrt{(\sigma/\pi)}$ instead of the radius R of the circle, where σ denotes the *area* of aperture.

It would be of interest to ascertain the electric capacity of a disc of nearly circular outline to the next approximation involving the square of $\delta\mathrm{R}$, the deviation of the radius in direction ω from the mean value. If $\delta\mathrm{R} = \alpha_n \cos n\omega$, α_1 would not appear, and the effect of α_2 is known from the solution for the ellipse. For other values of n further investigation is required.

In the case of the ellipse elongated apertures are not excluded, provided of course that the longer diameter is small enough in comparison with the diameter of the sphere. When e is nearly equal to unity,

$$\mathrm{F}(e) = \log\frac{4}{\sqrt{(1-e^2)}} = \log\frac{4\mathrm{R}_1}{\mathrm{R}_2}, \tag{62}$$

R_2 being the semi-axis minor. The pitch of the resonator is now comparatively independent of the small diameter of the ellipse, the large diameter being given.